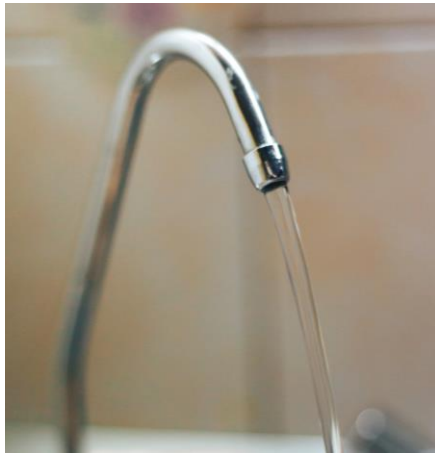
Free Shear Flows - Jets

Real External Flows – Lesson 5

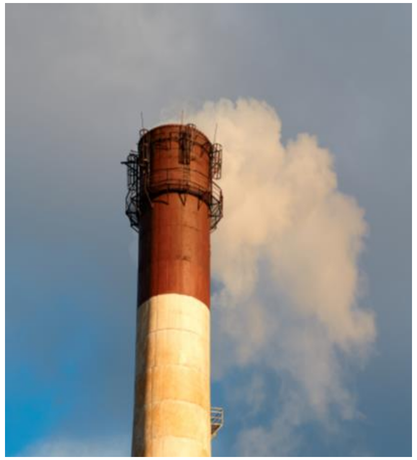
Free Shear Flows

- Most external flows focus on flows which are constrained by wall boundaries.
- In this lesson we will consider laminar free shear flow unaffected by walls which develop and spread in an open ambient fluid.
- The first type of free shear flows we will be discussing are jets.
- Jets are flow phenomena that exhaust from a confined source such as a nozzle into a uniform environment.
- Some common examples of jets include:
  - Exhaust from a rocket or a jet engine
  - Smoke coming out of a chimney
  - Water flowing out of a faucet
  - Exhaust from an automobile



Water from Tap

Rocket Exhaust

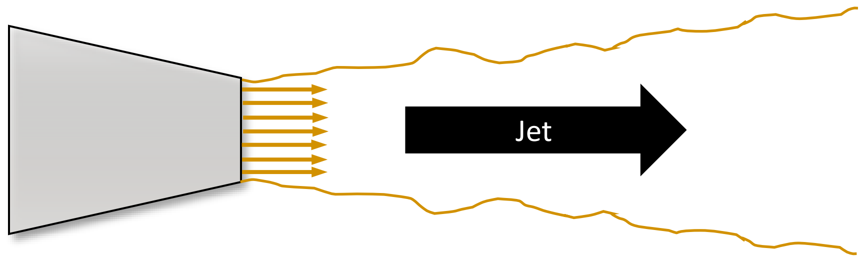


Smoke from chimney

Automobile Exhaust

Analysis Objective

- Jets can be analyzed using common parameters and assumptions:
  - Characteristic velocity scale $\bar{u}_{max}(x)$
  - Characteristic shear layer width $b(x)$
  - Near constant pressure throughout the flow since the flow is unconfined (assuming incompressible flow)
- Our analysis will focus on the asymptotic behavior of:
  - Width of the jet
  - Velocity scales
  - Velocity profiles:
    - $\bar{u}(x, y)$ - planar flow
    - $\bar{u}(x, r)$ - axisymmetric flow



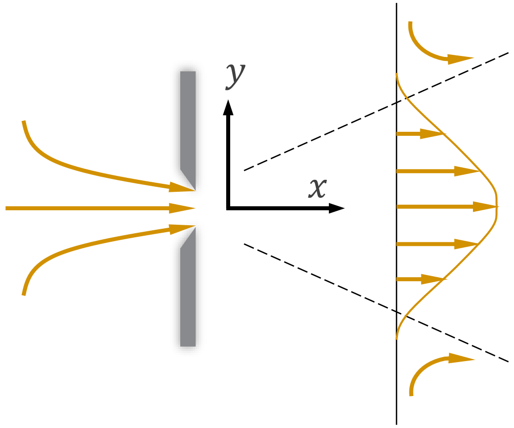
Free Shear Flow Equations

- Assuming a free-shear flow is dominated by the velocity in x-direction, and the Reynolds number is large, then the boundary-layer-like approximations hold:

$$v \ll u, \qquad \partial u/\partial x \ll \partial u/\partial y\,, \qquad \partial p/\partial y \approx 0, \qquad \partial p/\partial x \approx 0$$

- Then the 2D plane form of free-shear equations satisfy:

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0, \qquad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$



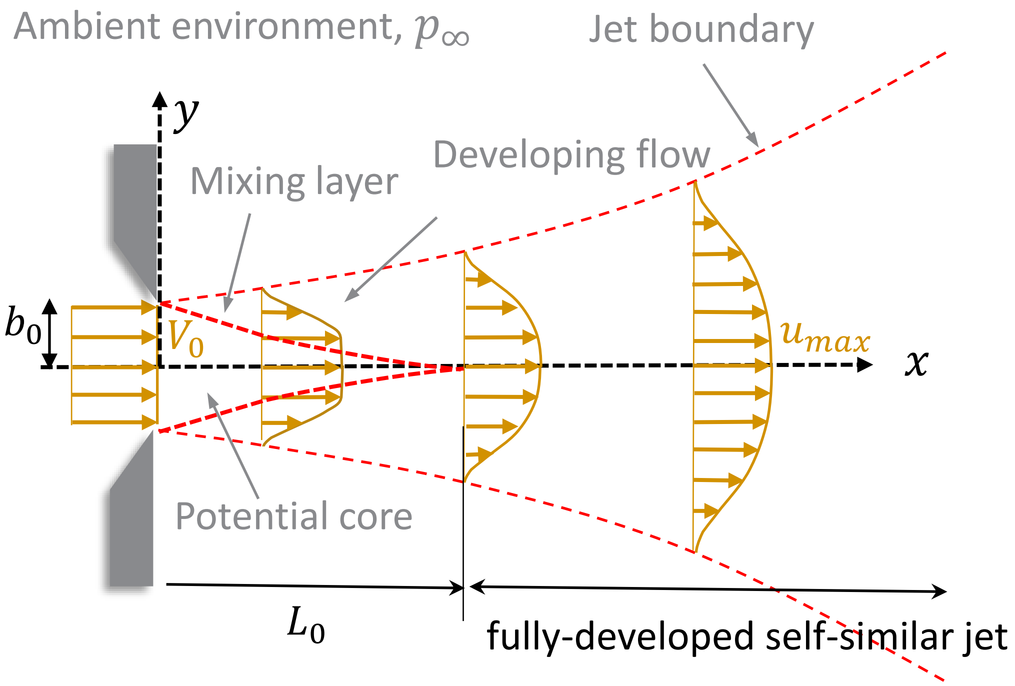
- The axisymmetric form of these equations is:

$$\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rv) = 0, \qquad u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right)$$

- This formulation holds in the fully developed self-similar region of the flow.
- We will be focusing on the solution in this self-similar region.

Plane Jet Flows

- Let's begin by examining a symmetric 2D plane jet.
- Flow in a channel of height $2b_0$ exits at a uniform velocity V_0 into a quiescent environment.
- Like a flat plate boundary layer, an initial sharp gradient in velocity (also known as a shear layer) is seen at the beginning of the jet just downstream of the opening between the ambient and nearly inviscid potential core.
- Velocity remains nearly constant through the potential core.
- As viscosity smooths out the gradients in velocity, the jet develops into a self-similar gaussian-like shape.
- Eventually at $L_0 \sim 40b_0$ the flow reaches a "fully-developed" condition and the velocity profile maintains a self-preserving shape:



The velocity profile preserves momentum but not mass as fluid is being entrained by ambient.

$$\frac{\bar{u}(x,y)}{\bar{u}_{max}(x)} \approx f\left(\frac{y}{b}\right)$$ or $$\frac{\bar{u}(x,r)}{\bar{u}_{max}(x)} \approx f\left(\frac{r}{b}\right)$$ axisymmetric

Plane Jet Momentum Flux

- We can obtain a solution for the fully-developed 2D plane jet by integrating the momentum equation in y in the bounding limits of $\pm\infty$.

$$\int_{-\infty}^{\infty} \rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) dy = \int_{-\infty}^{\infty} \frac{\partial \tau_T}{\partial y} dy$$

- Using integration by parts and using the continuity equation, this equation reduces to the simple integral:

$$M = \int_{-\infty}^{\infty} \rho\bar{u}^2 dy = \text{constant}$$

- Using the inlet boundary condition (constant inlet velocity profile), the constant can be evaluated:

$$\int_{-\infty}^{\infty} \rho\bar{u}^2 dy = 2b_0\rho V_0^2$$

- We see that for the plane jet, the total integrated momentum is a constant and independent of axial position. Note this holds for laminar and turbulent jets.

Laminar Plane Jet

- The analysis of a plane laminar jet can be carried out similar to that of the Blasius layer. Schlichting (1933) proposed the following similarity expression for the stream function:

$$\psi = \nu^{1/2} x^{1/3} f(\xi), \qquad \xi = \frac{y}{3\nu^{1/2} x^{2/3}}, \qquad u = \frac{f'(\xi)}{3x^{1/3}}, \qquad v = \frac{-\nu^{1/2}}{3x^{2/3}}(f - 2f'\xi)$$

- Substituting into the governing equations of the jet flow gives:

$$f''' + ff'' + f'^2 = 0, \qquad f(0) = f''(0) = f'(\infty) = 0$$

which, unlike the equation for the Blasius layer, has a simple analytical solution:

$$f(\xi) = 2a\tanh(a\xi) \qquad f'(\xi) = 2a^2 \operatorname{sech}^2(a\xi)$$

where the constant a is found by evaluating the integral of the momentum flux $M = \int_{-\infty}^{\infty} \rho u^2 dy$:

$$a = \left(\frac{9M}{16(\rho\mu)^{1/2}}\right)^{1/3} \approx 0.8255 \frac{M^{1/3}}{(\rho\mu)^{1/6}}$$

Laminar Plane Jet (cont.)

- The maximum centerline velocity is then given by:

$$u_{max} = \frac{2a^2}{3x^{1/3}} \approx 0.4543 \left(\frac{M^2}{\rho\mu x}\right)^{1/3}$$

As the jet spreads, the centerline velocity decays as $x^{-1/3}$

- The velocity distribution is:

$$u(x,y) = u_{max}\text{sech}^2(a\xi) = u_{max}\,\text{sech}^2\left[0.2752\left(\frac{M\rho}{\mu^2 x^2}\right)^{1/3} y\right]$$

- Jet width, defined as the twice the distance where $u = 0.01u_{max}$, is:

$$W \approx 21.8\left(\frac{x^2\mu^2}{M\rho}\right)^{1/3}$$

giving the jet spread rate as $x^{2/3}$

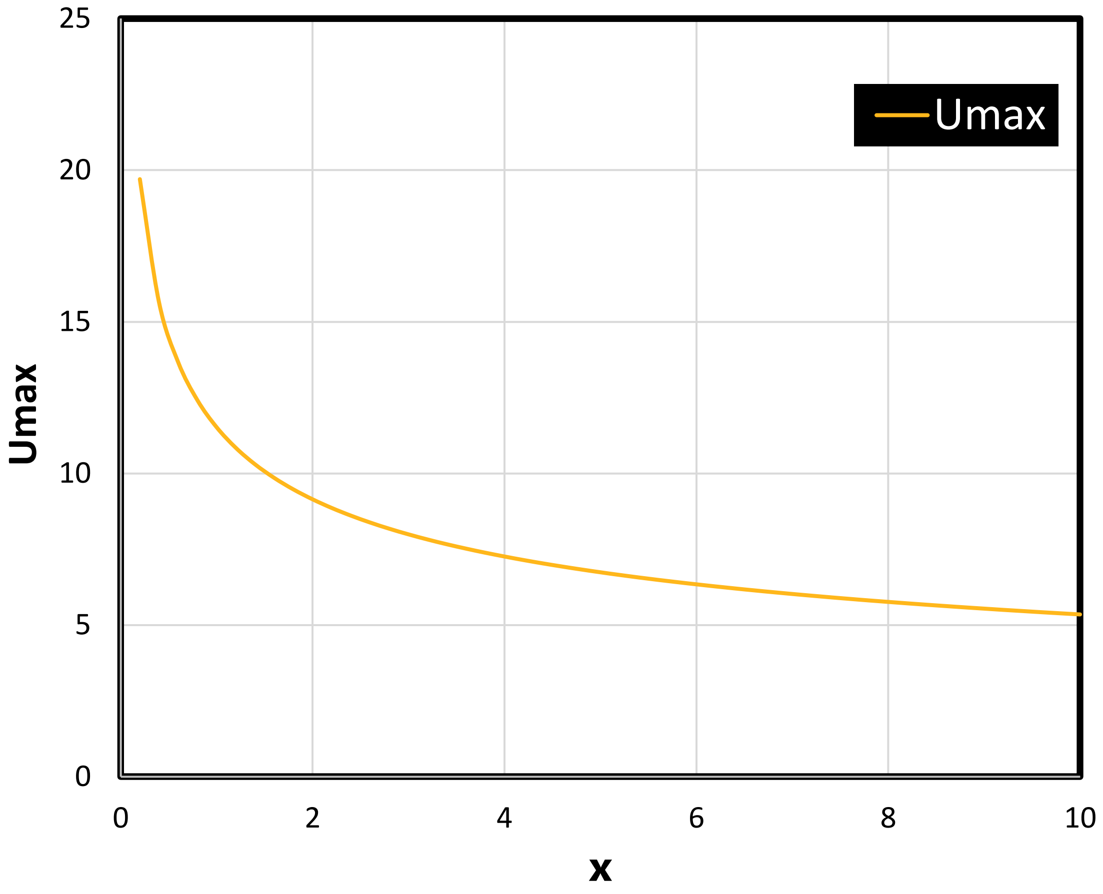
- The mass flow rate across an arbitrary vertical plane is:

$$\dot{m} = \rho \int_{-\infty}^{\infty} u dy = (36M\rho\mu x)^{1/3}$$

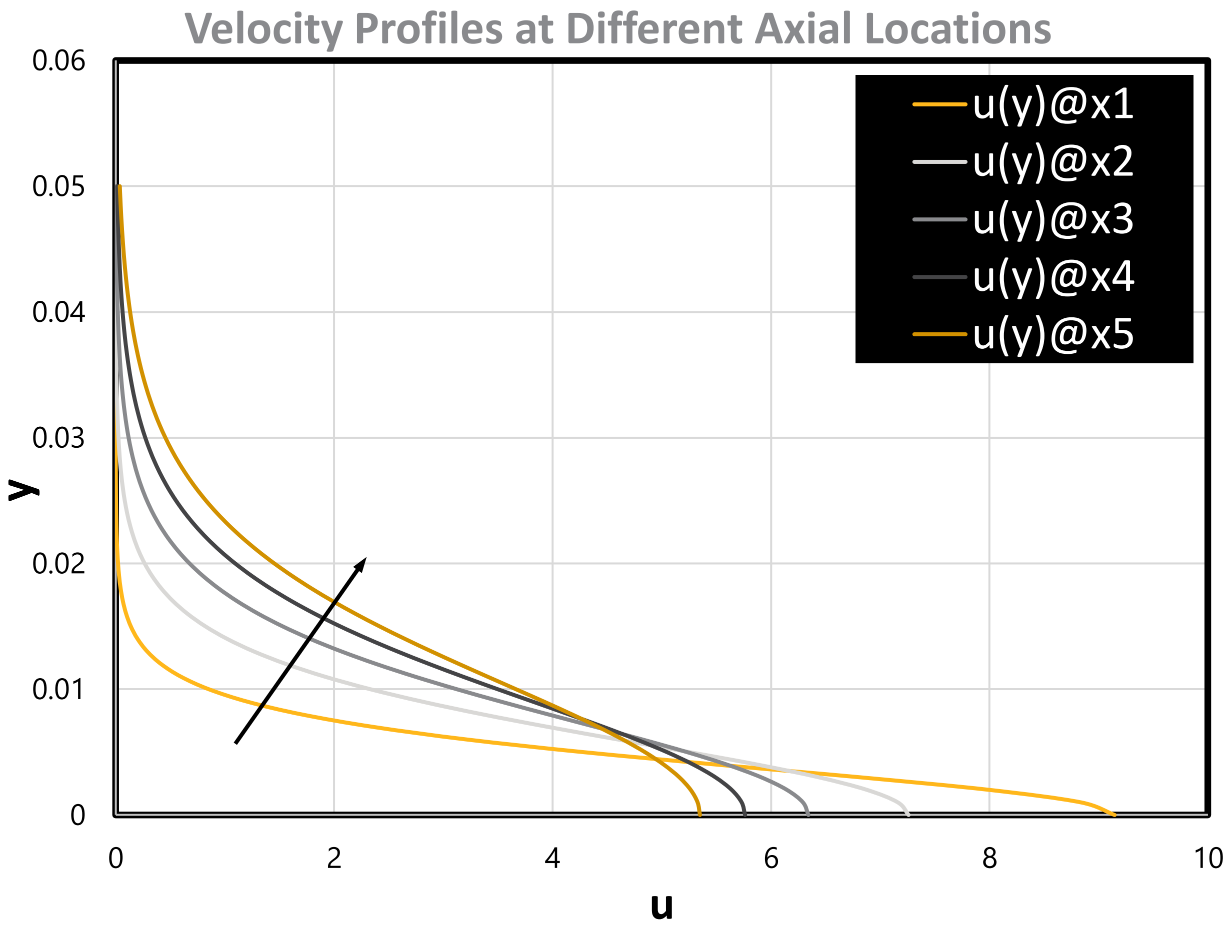
Mass flow rate increases as $x^{1/3}$ as the jet entrains fluid from ambient.

Plane jets undergo transition early at the Reynolds number $Re = 30$ based on exit slot width and exit velocity.

Laminar Plane Jet (cont.)



Jet centerline velocity ($\overline{u}_{max}$)



Velocity profiles

Self-Similarity of Turbulent Jets

- In the fully developed self-similar region, the jet momentum is conserved at each cross-section:

$$M = \int_{-\infty}^{\infty} \rho \bar{u}^2 dA = \text{constant}\, \rho b^n\, \bar{u}_{max}^2$$

$$n = \begin{cases} 1 & \text{plane jet} \\ 2 & \text{axisymmetric} \end{cases}$$

$$\bar{u}_{max} = \bar{u}(x, 0)$$

- The centerline velocity and jet width are functions of momentum, density and distance only. There is no dependence on the molecular viscosity in the absence of walls. Dimensional analysis of the momentum equations gives:

$$b = \text{constant}\ \ x$$

$$\bar{u}_{max} = \text{constant} \left(\frac{M}{\rho}\right)^{1/2} x^m$$

$$m = \begin{cases} -1/2 & \text{plane jet} \\ -1 & \text{axisymmetric} \end{cases}$$

- The constant in the above expressions is unique and independent of the Reynolds number. It is defined from experiment.

Turbulent Plane Jet Velocity Model

- A steady-state plane jet issuing into a quiescent environment can be modeled in a fashion similar to a boundary layer. We assume no axial pressure gradient and the velocity gradients are primarily affected by turbulent diffusion.

- Using appropriate scaling arguments, the governing equations and boundary conditions in 2D can be written as follows:

$$\frac{\partial \bar{u}}{\partial x} + \frac{\partial \bar{v}}{\partial y} = 0$$

$$\rho\left(\bar{u}\frac{\partial \bar{u}}{\partial x} + \bar{v}\frac{\partial \bar{u}}{\partial y}\right) = \frac{\partial \tau}{\partial y}, \qquad \tau = \mu_T \frac{\partial \bar{u}}{\partial y}$$

Boundary conditions:

$$\left.\begin{aligned} &\partial \bar{u}/\partial y|_{y=0} = 0 \\ &\bar{v}(x,0) = 0 \end{aligned}\right\} \text{symmetry}$$

$$\bar{u}(0,y) = V_0$$

$$\bar{u}(x,y) = 0 \quad y \to \pm\infty$$

$$\bar{\tau}(x,y) = 0 \quad y \to \pm\infty$$

- Prandtl (1926) suggested the turbulent viscosity μ_T is a function of x only:

$$\mu_T \approx \kappa \rho b \bar{u}_{max} = \text{constant}\; x^{1/2}, \quad \kappa = 0.016$$

from dimensional analysis

Plane Turbulent Jet Solution Using Modeled Eddy Viscosity

- Görtler (1942) proposed the following similarity variables:

$$\bar{u} = V_0\left(\frac{x_0}{x}\right)^{1/2} f'(\xi), \qquad \nu_T = \kappa V_0 b_0 \left(\frac{x_0}{x}\right)^{1/2}, \qquad \xi = \frac{\sigma y}{x}, \qquad \sigma - \text{constant}$$

- Substitution into continuity and momentum equations gives a similarity ODE for f:

$$\frac{1}{2} f''' + f f'' + f'^2 = 0, \qquad f(0) = f''(0) = f'(\infty) = 0$$

- The solution is given by:

$$f = \tanh(\xi), \qquad \frac{\bar{u}}{\bar{u}_{max}} = \operatorname{sech}^2(\xi), \qquad \sigma \approx 7.67 \text{ (from experiment)}$$

- The width of the jet cannot be clearly defined since $\bar{u}(y \to \infty) \to 0$. Defining half-velocity point $y_{1/2}$ where $\bar{u} = \bar{u}_{max}/2$, the jet growth can be estimated as:

$$\frac{2y_{1/2}}{x} = \frac{1.76}{7.67} = \tan 13^{\circ}$$

which is independent of Reynolds number

Plane Turbulent Jet Solution Using Assumed Velocity Profile

- A similarity solution can also be developed using assumed velocity profile. Taking the profile defined by the similarity variable as:

$$\bar{u} = \bar{u}_{max}\, f(\xi), \qquad \xi = y/x$$

and assuming the velocity profile is shaped like a Gaussian curve, the following mathematical representation can be defined:

$$\frac{\bar{u}}{\bar{u}_{max}} = f(\xi) = e^{-\frac{\xi^2}{2C_1^2}}$$

- Substitution into the jet integral equations yields the following expression for the centerline velocity $\bar{u}_{max}$

$$\frac{\bar{u}_{max}}{V_0} = \sqrt{\frac{2b_0}{I_2 x}}, \qquad I_2 = \int_{-\infty}^{\infty} f^2(\xi)\, d\xi$$

C_1 is a constant to be determined from experimental data

Plane Turbulent Jet Solution Using Assumed Velocity Profile (cont.)

- L_0 can be computed by noting that $\bar{u}_{max}/V_0 = 1$ at $x = L_0$:

$$\boxed{L_0 = \frac{2b_0}{I_2}}$$

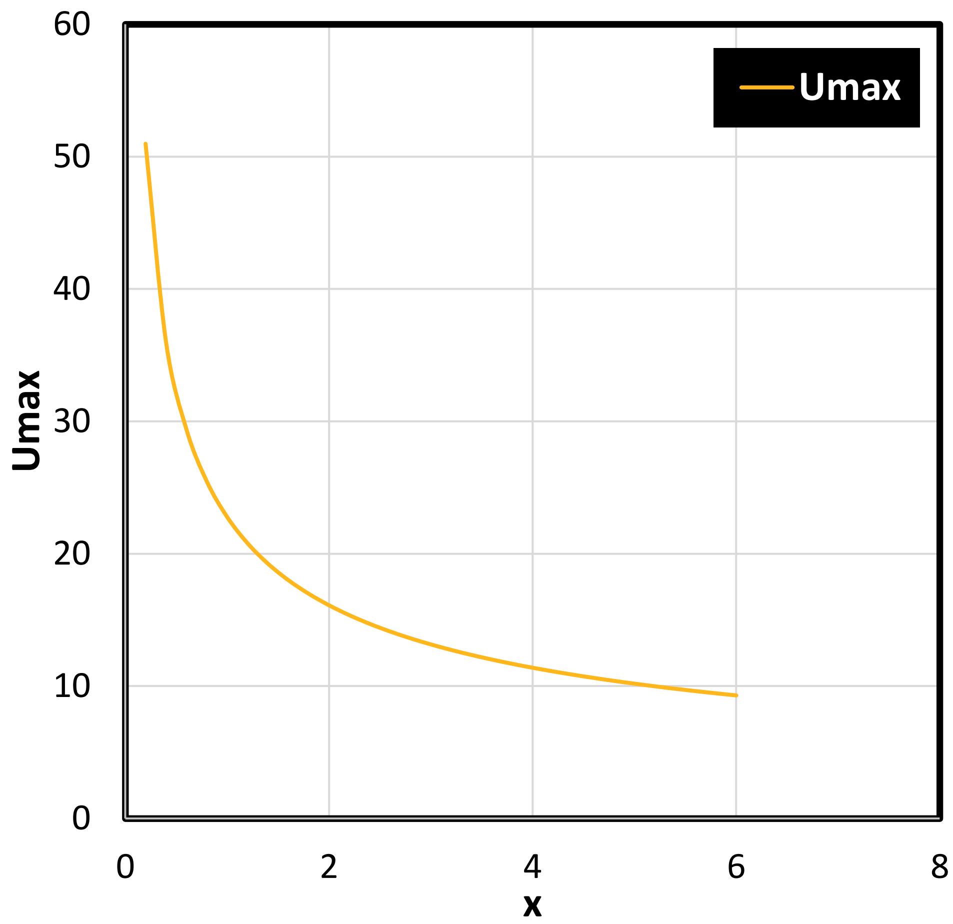
- The integral of $f^2(\xi)$ and hence I_2 can be determined once C_1 is known. From experiments by Albertson et al. (1950), it is found that a value of $C_1 = 0.109$ provides a good fit to the data, whereby $I_2 = 0.272$, and the following solution is obtained:

$$\frac{\bar{u}}{\bar{u}_{max}} = e^{-42.084(y/x)^2}$$

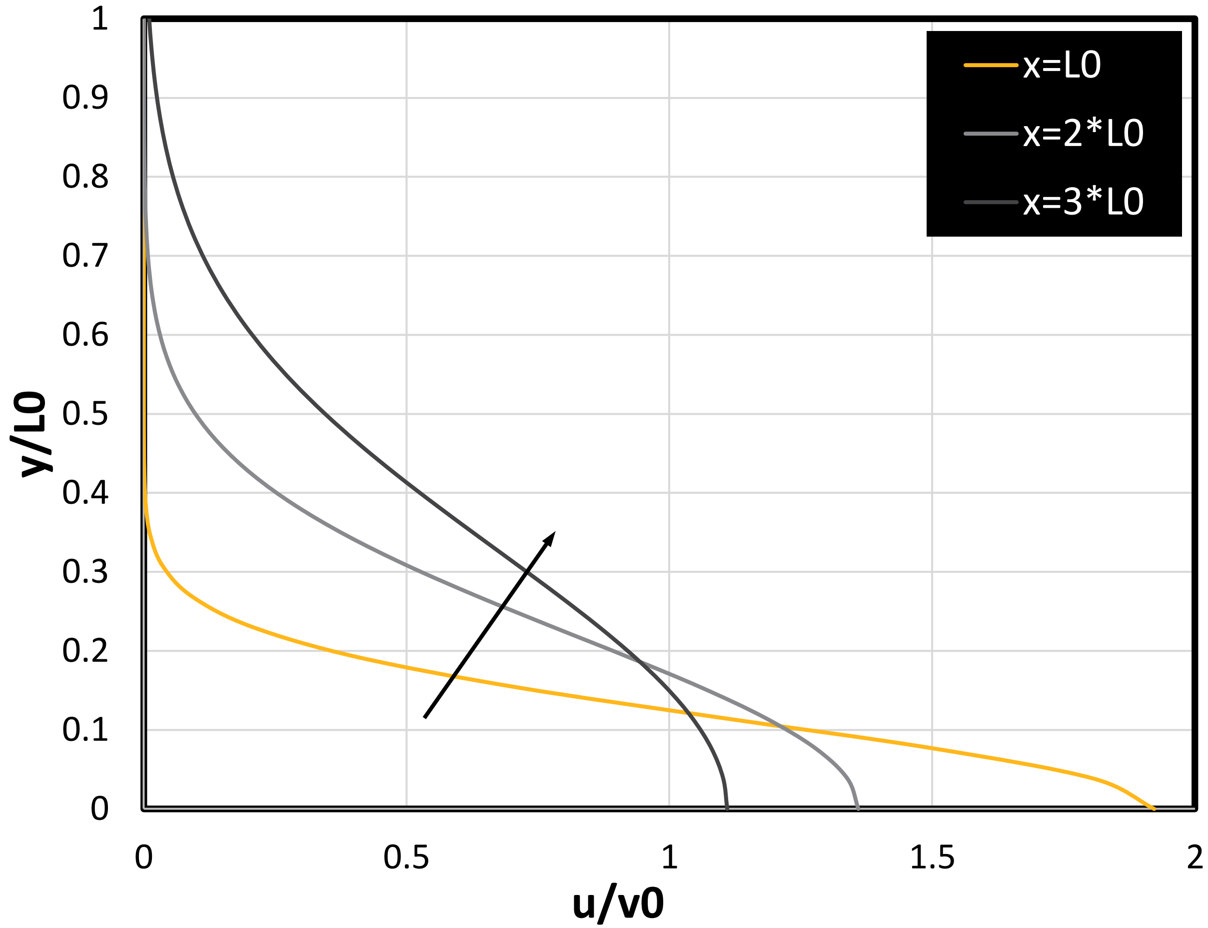
$$\frac{\bar{u}_{max}}{V_0} = 3.224\sqrt{\frac{b_0}{x}}$$

$$L_0 = 10.4b_0$$

Plane Jet Velocity Profiles and $\bar{u}_{max}$ ($b_0 = 0.5m$, $V_0 = 10$ m/s)



Jet centerline velocity ($\bar{u}_{max}$)



Velocity profiles

Laminar Narrow Round Jet Solution

- Consider a round jet emerging from a circular hole with enough momentum that it remains narrow and grows slowly.
- In such cases, the radial changes $\partial/\partial r$ are much larger than axial changes $\partial/\partial x$ – *Narrow Jet*
- Schlichting showed the jet thickness grew linearly and the similarity variable was r/x, and appropriate stream function is:

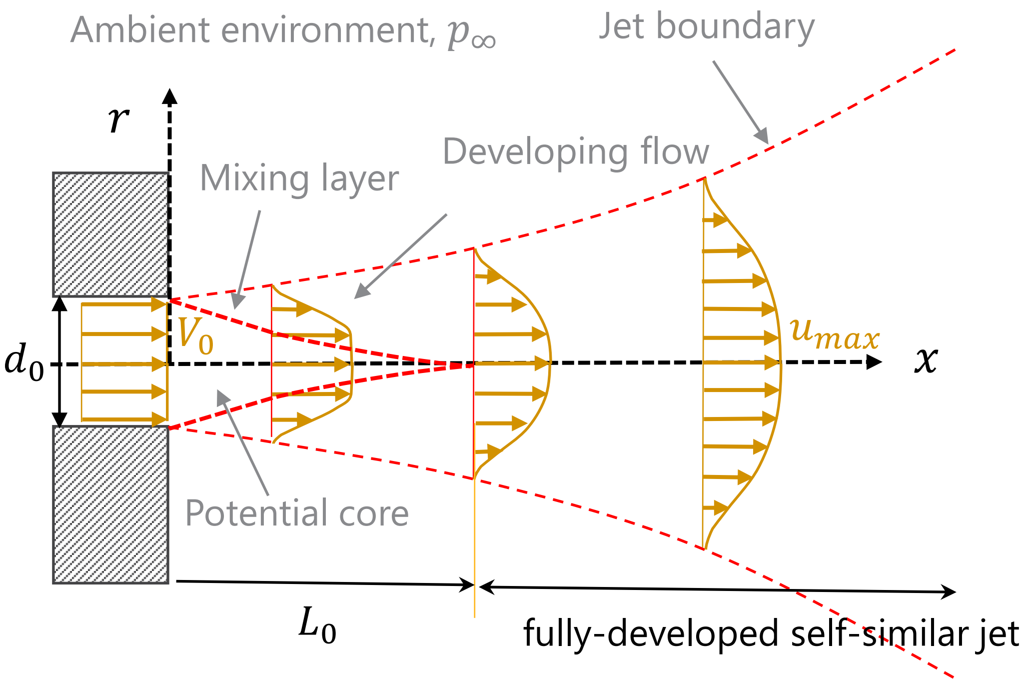
$$\psi(r,x) = \nu x f(\xi), \qquad \xi = r/x$$

- The velocity components are then given by:

$$u = \frac{1}{r}\frac{\partial\psi}{\partial r} = \frac{\nu f'}{r}, \qquad v = -\frac{1}{r}\frac{\partial\psi}{\partial x} = \frac{\nu}{r}(\xi f' - f)$$

- Substitution into the momentum equation yields the similarity ODE:

$$\frac{\partial}{\partial\xi}\left(f'' - \frac{f'}{\xi}\right) = \frac{1}{\xi^2}(ff' - \xi f'^2 - \xi f f'')$$



$$\underbrace{f(0) = f'(0) = f'(\infty) = 0}_{\text{Boundary conditions}}$$

Laminar Round Jet Solution (cont.)

- This equation, despite its apparent complexity, has an exact solution:

$$f = \frac{(C\xi)^2}{1 + (C\xi/2)^2}$$

where the constant is found from the momentum of the jet in a manner similar to that of the plane jet:

$$M = \rho \int_0^\infty u^2 2\pi r dr = \frac{16\pi}{3}\rho C^2 \nu^2, \qquad C = \left(\frac{3M}{16\pi\rho\nu^2}\right)^2$$

- The axial jet velocity:

$$u = \frac{3M}{8\pi\mu x}\left(1 + \frac{C^2\xi^2}{4}\right)^{-2}$$

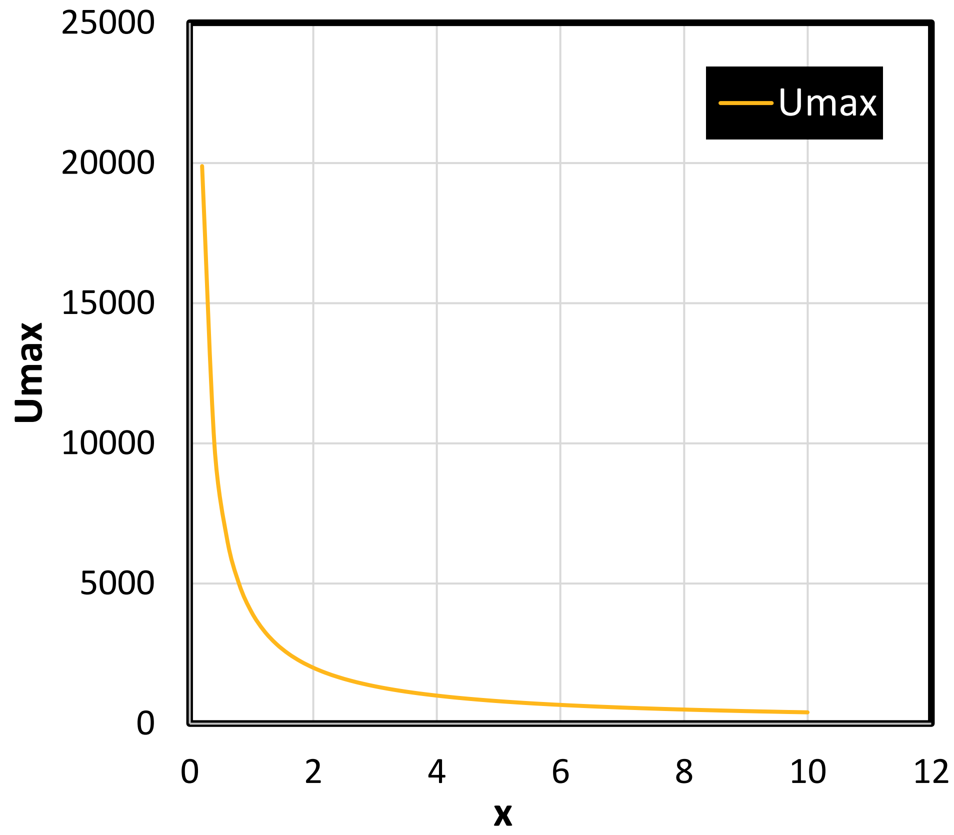
- The mass flow rate across any cross-section is:

$$\dot{m} = \rho \int_0^\infty u 2\pi r dr = 8\pi\mu x$$

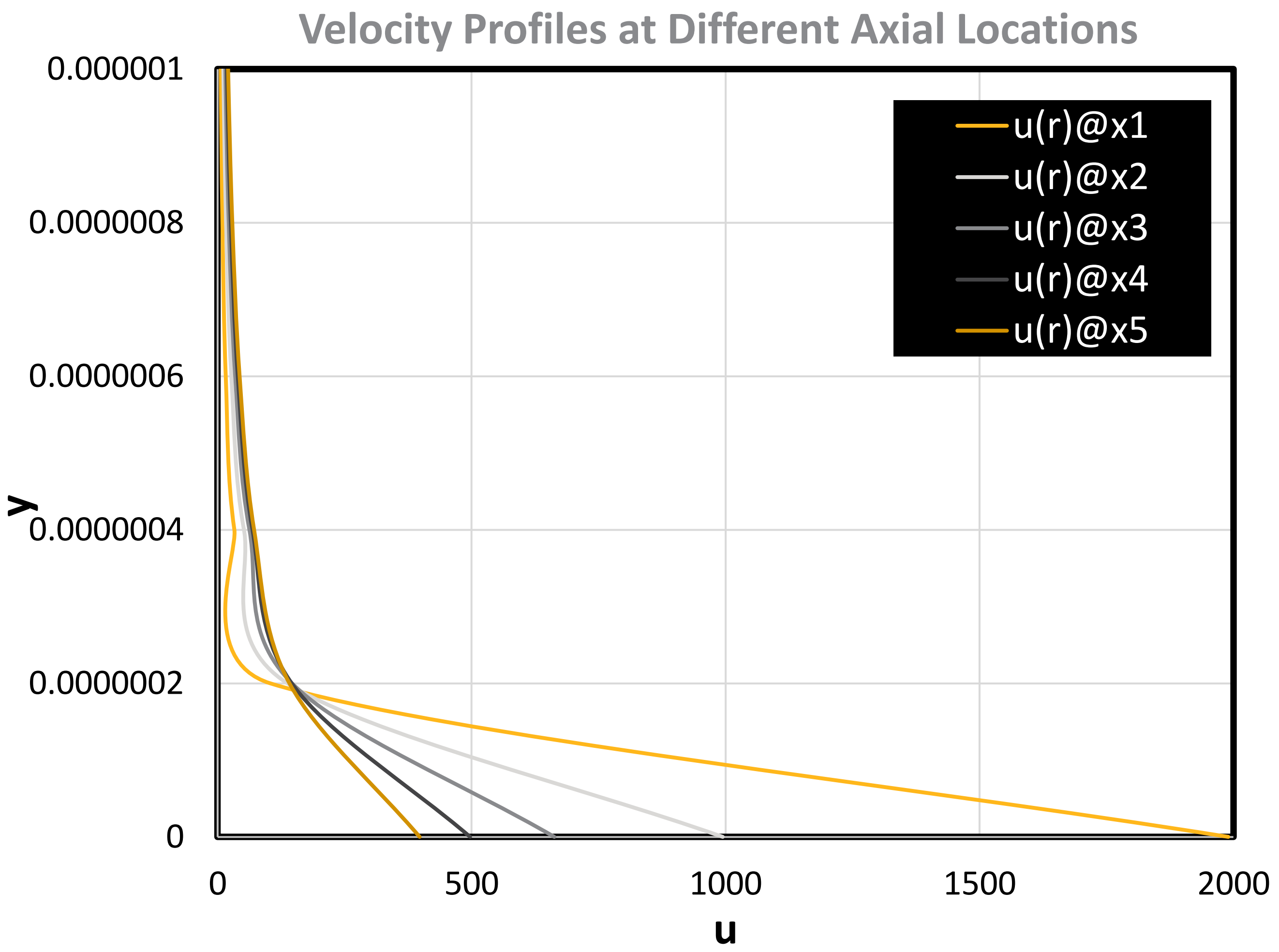
which, unlike the plane jet, is independent of the jet momentum.

- Round jets are much more resilient to instabilities than plane jets and they transition to turbulence at $Re = V_0 d_0/\nu \sim 2{,}000$

Laminar Round Jet Solution (cont.)



Jet centerline velocity ($\overline{u}_{max}$)



Turbulent Round Jet Solution

- The round jet can be analyzed in a manner similar to the plane jet, except we assume an axisymmetric jet and employ a cylindrical-polar form of the 2D equations.

- From self-similarity analysis, jet diameter increases linearly with x, and $\bar{u}_{max}$ decreases inversely with x, thus the eddy viscosity is constant throughout the mixing region of the jet. Kinematic eddy viscosity from experimental fit:

$$\nu_T \approx 0.00196 x \bar{u}_{max}$$

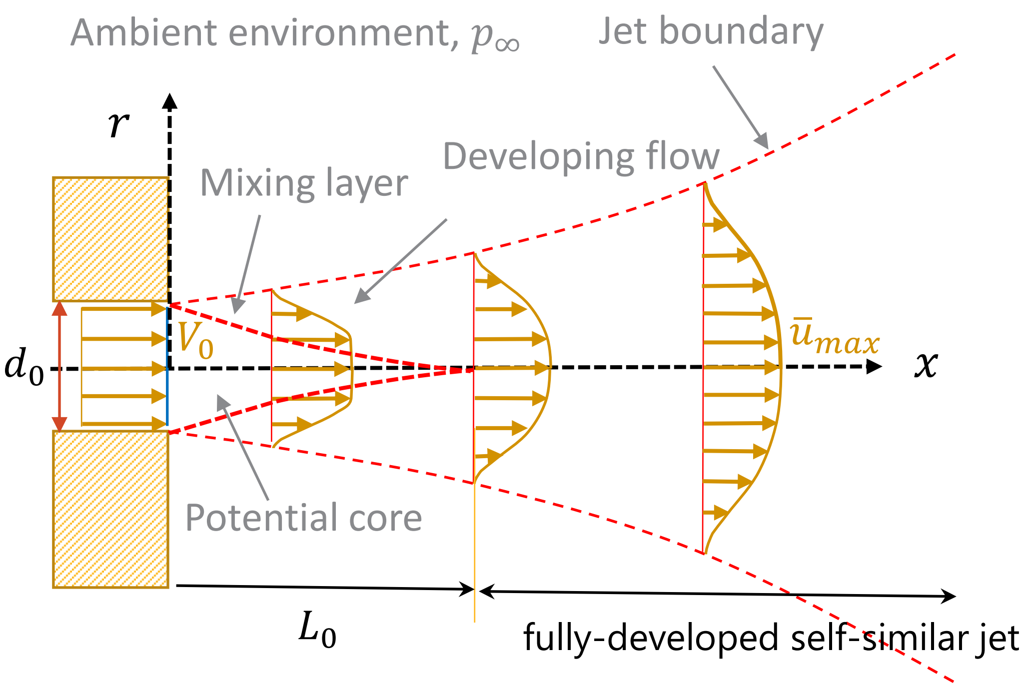
- The solution for the velocity distribution is:

$$\frac{\bar{u}}{\bar{u}_{max}} = \left(1 + \bar{u}_{max} \frac{r^2}{8\nu_T x}\right)^{-2} = \left(1 + 62.5 \frac{r^2}{x^2}\right)^{-2}$$

Note it has the same form as the solution for laminar round jet.

- Distribution of the centerline velocity is:

$$\bar{u}_{max}/V_0 = 6.4(d_0/x)$$



- The constant kinematic eddy viscosity is:

$$\nu_T = 0.013 V_0 d_0$$

$$\Downarrow$$

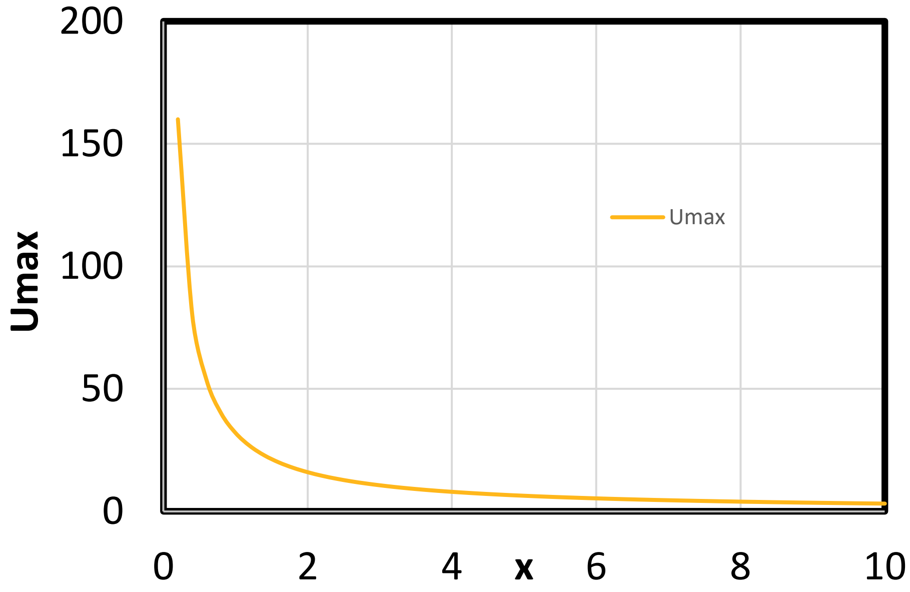
$$\nu_T/\nu = 0.013(V_0 d_0/\nu) = 0.013 Re$$

Turbulent viscosity ratio

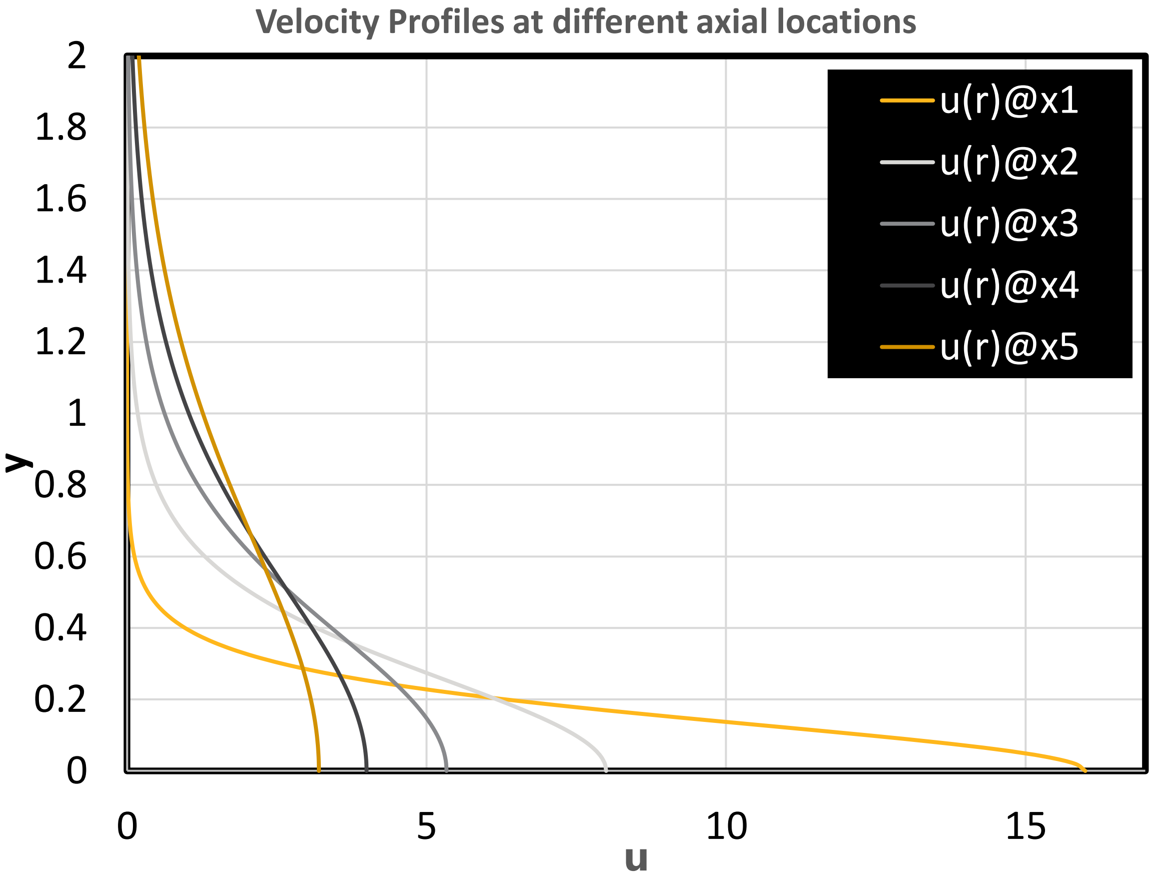
Turbulent Round Jet Solution

- It turns out a somewhat better correlation with experimental data can be achieved by adopting the plane jet solution with a different constant σ:

$$\frac{\bar{u}}{\bar{u}_{max}} = \mathrm{sech}^2\left(10.4\frac{r}{x}\right)$$



Jet centerline velocity $(\bar{u}_{max})$



Velocity profiles

Summary

- In this lesson, we discussed the significance of both plane and round jets.
- We obtained correlations to predict jet spreading rates using similarity solutions for both laminar and turbulent jets.
- While laminar jets are relatively easy to analyze, turbulent jet solutions require experimental results to obtain certain empirical parameters.
- These correlations are used to benchmark and validate high-fidelity CFD codes even today.

Ansys