Governing Equations of Turbulent Flows

Basics of Turbulent Flows – Lesson 4

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Intro

- In this lesson, we will discuss the concept of time averaging which has been relied upon in developing turbulence models, especially for practical engineering flows, since the time of Reynolds.
- The idea behind time averaging is to:
  - Separate flow variables into mean flow and turbulent fluctuating components.
  - Apply time averaging to the equations to reduce instantaneous turbulent fluctuations to their time-averaged values.
  - Somehow find a way to describe time-averaged fluctuation terms (turbulence modeling).

Time-Averaged Flow Field

- Due to the random nature of turbulent fluctuations, it is useful to describe them using statistical measures. To this end, let us introduce the principles of time averaging.
- Using the decomposition shown on the previous slide, any flow variable in an unsteady turbulent flow can be expressed as the sum of a mean and time varying components.
- For an arbitrary variable ϕ we can write:

$$\phi = \bar{\phi} + \phi'$$

where $\bar{\phi}$ = mean component and ϕ' = fluctuating component

- The mean is defined by a time average over a representative sampling period T:

$$\bar{\phi} = \frac{1}{T}\int_0^T \phi \, dt$$

- The sampling period is assumed to be long enough such that the random fluctuations average to zero:

$$\overline{\phi'} = \frac{1}{T}\int_0^T \phi' \, dt = 0$$

Rules of Time-Averaging

- For sums and products of two flow variables ϕ_1, ϕ_2, it can be shown that the following rules hold:

$$\overline{\overline{\phi_1}\phi_2'} = 0$$
$$\overline{\overline{\phi_1}\phi_2} = \overline{\phi_1}\,\overline{\phi_2}$$
$$\overline{\phi_1 + \phi_2} = \overline{\phi_1} + \overline{\phi_2}$$
$$\overline{\phi_1\phi_2} = \overline{\phi_1}\,\overline{\phi_2} + \overline{\phi_1'\phi_2'}$$

- Note that, in general, $\overline{\phi_1'\phi_2'} \neq 0$ unless the two variables are uncorrelated (that is, the fluctuations of ϕ_1' are completely independent of ϕ_2').
- Time averages of spatial derivatives are the spatial derivatives of the time-averaged variables:

$$\overline{\frac{\partial \phi}{\partial x}} = \overline{\frac{\partial\left(\overline{\phi}+\phi'\right)}{\partial x}} = \frac{\partial \overline{\phi}}{\partial x} + \overline{\frac{\partial \phi'}{\partial x}}^{\,0} = \frac{\partial \overline{\phi}}{\partial x}$$
$$\overline{\frac{\partial^2 \phi}{\partial x^2}} = \overline{\frac{\partial^2\left(\overline{\phi}+\phi'\right)}{\partial x^2}} = \frac{\partial^2 \overline{\phi}}{\partial x^2} + \overline{\frac{\partial^2 \phi'}{\partial x^2}}^{\,0} = \frac{\partial^2 \overline{\phi}}{\partial x^2}$$

Reynolds Averaging - Intro

- We will now show how the Reynolds-Averaged Navier Stokes (RANS) equations are derived and how turbulence models are developed for them. For simplicity, we will consider the incompressible Navier-Stokes equations (density and viscosity are assumed constant). More general versions can be derived with similar techniques, for example, ones that apply to compressible flows.
- The basic procedure is as follows:
  - Introduce the flow variables represented as mean + fluctuating components into a given equation.
  - Time average the equation. From the rules of time averaging defined earlier, the time average of the equation is simply the time average of the individual terms.
  - Eliminate the terms involving averages of individual fluctuation quantities (which are zero) and write the results in terms of the averaged terms + terms involving averages of fluctuating combinations of terms (which are not in general zero).
- The goal of Reynolds Averaging is to develop a set of equations with which we can solve for the mean flow field. Since the mean flow field is often steady, we can develop steady-state solution approaches for the mean flow that require much less computational effort and resources than solving for the unsteady flow.

Reynolds-Averaged Continuity Equation

- To demonstrate Reynolds Averaging, we will consider the incompressible Navier-Stokes equations with no body forces:

$$\nabla \cdot \vec{V} = 0$$

$$\rho \left[\frac{\partial \vec{V}}{\partial t} + \vec{V} \cdot \nabla \vec{V} \right] = -\nabla p + \mu \nabla^2 \vec{V}$$

- Substituting the decomposed velocity components into the continuity equation gives

$$\nabla \cdot \left[(\bar{u} + u')\hat{\imath} + (\bar{v} + v')\hat{\jmath} + (\bar{w} + w')\hat{k} \right] = 0$$

- Upon time averaging, we see that the average of the fluctuating velocities vanish, leaving

$$\overline{\nabla \cdot \vec{V}} = \nabla \cdot \left(\bar{u}\hat{\imath} + \bar{v}\hat{\jmath} + \bar{w}\hat{k} + \overset{0}{\overline{u'}}\hat{\imath} + \overset{0}{\overline{v'}}\hat{\jmath} + \overset{0}{\overline{w'}}\hat{k} \right) = 0$$

Here we define the mean velocity vector as $\vec{\bar{V}} = \bar{u}\hat{\imath} + \bar{v}\hat{\jmath} + \bar{w}\hat{k}$

- Thus the RANS continuity equation is identical in form to the original continuity equation.

$$\nabla \cdot \vec{\bar{V}} = 0$$

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Reynolds-Averaged Momentum Equations

- Applying this technique to the momentum equations, we obtain:

$$\rho\left[\frac{\partial\vec{\bar{V}}}{\partial t}+\vec{\bar{V}}\cdot\nabla\vec{\bar{V}}\right]+\rho\frac{\partial}{\partial x_j}\left(\overline{u_i'u_j'}\right)=-\nabla\bar{p}+\mu\nabla^2\vec{\bar{V}}$$

- Note the appearance of the additional term $\overline{u_i'u_j'}$ representing the turbulent inertia tensor. This term cannot be neglected in a turbulent flow and it is unknown.
- Considering that this term is a tensor, our time-averaging procedure introduced nine(!) new variables which need to be somehow described.

Reynolds-Averaged Momentum Equations (cont.)

- The averaged momentum equation can be re-arranged as:

$$\rho\left[\frac{\partial\vec{\bar{V}}}{\partial t}+\vec{\bar{V}}\cdot\nabla\vec{\bar{V}}\right]=-\nabla\bar{p}+\nabla\cdot\tau_{ij}$$

where

$$\tau_{ij}=\underbrace{\mu\left(\frac{\partial\bar{u}_i}{\partial x_j}+\frac{\partial\bar{u}_j}{\partial x_i}\right)}_{\text{laminar}}\underbrace{-\rho\overline{u_i'u_j'}}_{\text{turbulent}}$$

- Thus, this additional term can be thought of as the turbulent stress tensor.
- Its off-diagonal components, $\overline{u_i'u_j'}, i \neq j$, are referred to as turbulent shears.
- Turbulent stress tensor is commonly known as the Reynolds Stress tensor, and its components are denoted as R_{ij}. They can be written in a tensor matrix form in (x, y, z) as:

$$\begin{bmatrix}\rho\overline{u'u'} & \rho\overline{u'v'} & \rho\overline{u'w'}\\ \rho\overline{v'u'} & \rho\overline{v'v'} & \rho\overline{v'w'}\\ \rho\overline{w'u'} & \rho\overline{w'v'} & \rho\overline{w'w'}\end{bmatrix}$$

RANS Closure Problem

- There are six additional unknowns (because of symmetric nature of the stress tensor) introduced into the governing equations by the averaging process and the problem becomes under-defined as the number of unknowns is greater than the number of equations.
- To make RANS problem well-defined, we need to either:
  - Derive six additional governing equations for the Reynolds stresses, or
  - Express stress tensor components in terms of flow variables to make the problem well-defined without the addition of extra equations
- This challenge is referred to as the Turbulent Closure Problem.
- Unlike viscous stresses in the Navier-Stokes equations, which were related to the strain rate through rigorous physical arguments, no such exacting physics-based approach exists for Reynolds stresses.
- The need to close RANS equations gave rise to the science (and art!) of turbulence modeling.
- There has been a tremendous amount of research over many decades attempting to develop general ways to address closure. Alas, today we have many turbulence models which have been shown to be reliable and reasonably accurate for a wide range of flows, but they are NOT universal.

Summary

- We have discussed Reynolds Averaging and have derived the Reynold Averaged Navier-Stokes (RANS) equations for incompressible flow.
- The resulting set of RANS equations is under-defined because of the arousal of nine unknown Reynolds stresses, and require additional modeling effort to close the problem.
- We will briefly describe some fundamentals of turbulence modeling in the next lesson.

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