Flat-Plate Blasius Solution

Laminar Boundary Layer Theory – Lesson 4

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The Blasius Solution

- The first person to obtain a solution to boundary layer equations was Paul Richard Heinrich Blasius (1883 - 1970), one of Prandtl's students at the University of Gottingen.
- He developed an analytical technique for solving the equations in his Ph.D. thesis, published in 1909.
- This work was groundbreaking given that general numerical solutions to the Navier-Stokes equations did not exist and most analytical solutions assumed ideal, inviscid flow.
- We will present Blasius' basic analysis for a flat plate, and then provide the essential results, including correlations for boundary layer thickness, displacement thickness and skin friction.
- His solution approach was later extended by Falkner and Skan to include additional effects and configurations.

Flat Plate Boundary Layer Governing Equations

- The core of Blasius' analysis centers upon transforming the partial differential equations (PDEs) which comprise the flat plate boundary layer equations, with zero pressure gradient, into a single ordinary differential equation (ODE) by using a coordinate transformation approach.

- As noted previously, for a flat plate the boundary layer equation reduces to the pair of PDEs, which we will attempt to solve for the velocity components, u and v.

- For this analysis, it is assumed that the properties are known and hence we know the Reynolds Number, Re_x.

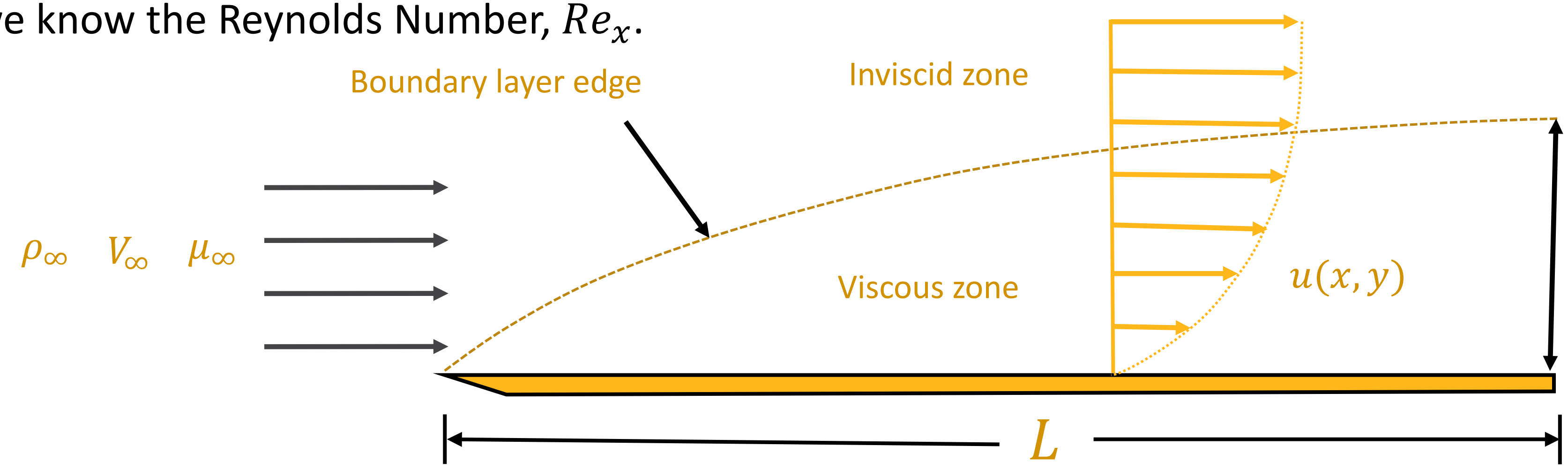
$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = \nu\frac{\partial^2 u}{\partial y^2}$$

$$u(x,0) = 0$$

$$v(x,0) = 0$$

$$u(x,y) \rightarrow V_\infty \ \ as\, y \rightarrow \infty$$



$$\delta(x) \sim \frac{x}{\sqrt{Re_x}}$$

$$Re_x = \frac{\rho_\infty V_\infty x}{\mu_\infty}$$

Stream Function

- The Blasius solution uses a stream function for the 2D flow, which by its definition satisfies the continuity equation. In terms of fluid velocities, we know the stream function as:

$$u = \frac{\partial \psi}{\partial y}, \qquad v = -\frac{\partial \psi}{\partial x}$$

Using stream function definitions for the velocity components in the x-momentum equation, we will seek to transform the PDE into an ODE, where the dependent variable is related to the stream function and the independent variable, which will be denoted η, is a function of x and y.

- The resulting solution is known mathematically as a self-similar solution, wherein the solution is essentially the same if the independent and dependent variables of the governing equations are appropriately scaled.

Similarity Transformation

- Recall that the boundary layer thickness correlates with x distance according to:

$$\delta(x) \sim \frac{x}{\sqrt{Re_x}}$$

- To obtain a similarity solution, Blasius first defined a non-dimensional y coordinate variable, η, based upon this boundary layer thickness correlation:

$$\eta(x,y) \sim \frac{y}{\delta(x)} = y\sqrt{\frac{V_\infty}{\nu x}}$$

- For the dependent variable function, denoted $f(\eta)$, Blasius used a scaled form of the stream function:

$$\psi = \sqrt{\nu V_\infty x} f(\eta)$$

- Physically, this corresponds to ψ growing proportional to δ with x.

The Blasius Boundary Layer Equation

- Substitution of the stream function into the momentum equations give the equation for f, which is the final form of the Blasius boundary layer equation for a flat plate:

$$f''' + \frac{1}{2} f f'' = 0$$

- The Boundary Conditions for the Blasius equation follow from the wall no-slip condition and requirement that wall forms a streamline ($f = 0$), and the asymptotic requirement that the velocity becomes the freestream velocity outside the boundary layer:

$$f(\eta = 0) = 0$$
$$f'(\eta = 0) = 0$$
$$f'(\eta \to \infty) = 1$$

Solution Methods for the Blasius Equation

- Blasius used an analytical series solution technique to solve his equations in his original work. With the availability of computers, we can now develop a numerical solution and calculate it with a high degree of accuracy.
- The numerical method for calculating the solution involves casting the Blasius equation as a set of coupled first order ODEs.

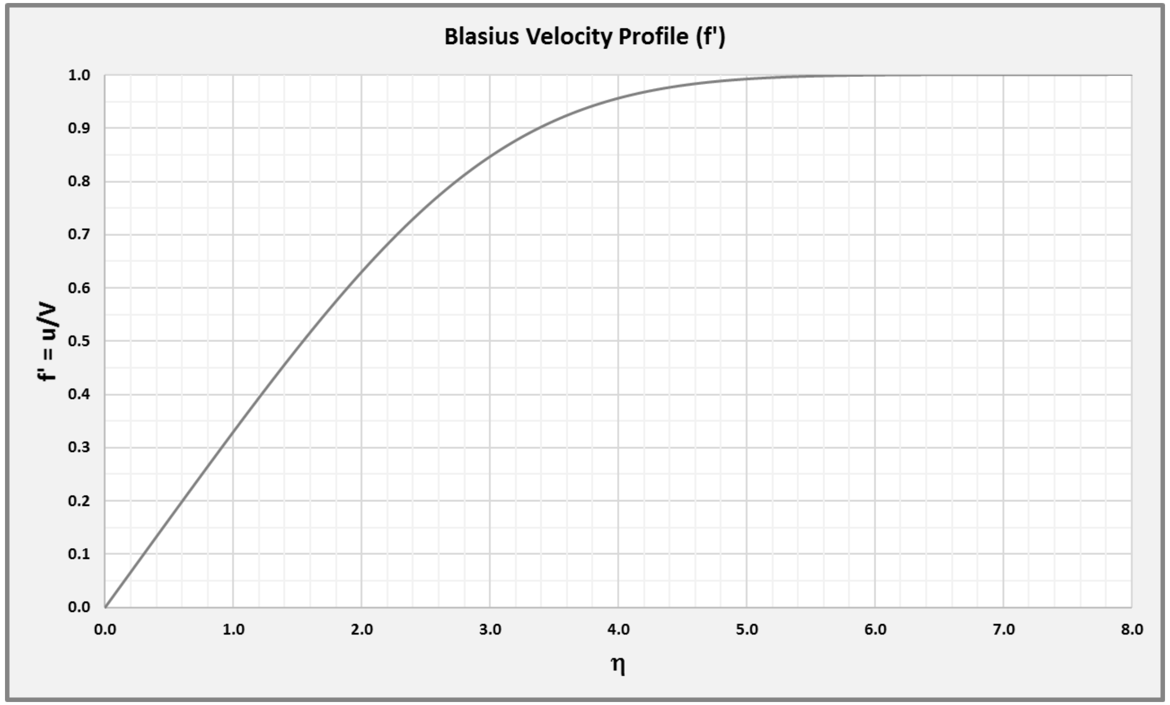
$$f'(\eta) = g(\eta)$$
$$g'(\eta) = f''(\eta) = h(\eta)$$
$$h'(\eta) = f'''(\eta) = -\frac{1}{2} f(\eta)h(\eta)$$

The boundary conditions give us $f(0) = f'(0) = g(0) = 0$, but we don't have a value for $h(0)$. We can employ an iterative approach called the "shooting algorithm" to determine which initial condition for h yields the asymptotic condition $f'(\eta \to \infty) = 1$. The algorithm proceeds as follows:

1. Guess a value for $h(0)$.
2. Numerically solve the coupled ODEs by marching in η, and halt when $f'(\eta) = g(\eta)$ has stopped changing (within a small error tolerance). Note the value of $f'(\eta)$. If it is > 1.0 then decrease $h(0)$, otherwise increase $h(0)$.
3. With the updated $h(0)$ (increased or decreased) repeat 2. Continue until $f'(\eta)$ is sufficiently close to 1.0.

Tabulated Blasius Solution

η	f	f'	f''
0.00	0.00000	0.00000	0.33200
0.20	0.00664	0.06641	0.33193
0.40	0.02656	0.13277	0.33142
0.60	0.05974	0.19894	0.33003
0.80	0.10611	0.26472	0.32735
1.00	0.16558	0.32980	0.32298
1.20	0.23796	0.39381	0.31657
1.40	0.32301	0.45631	0.30785
1.60	0.42037	0.51683	0.29666
1.80	0.52959	0.57486	0.28294
2.00	0.65013	0.62989	0.26676
2.20	0.78134	0.68147	0.24836
2.40	0.92249	0.72918	0.22810
2.60	1.07278	0.77269	0.20646
2.80	1.23132	0.81178	0.18400
3.00	1.39724	0.84634	0.16134
3.20	1.56963	0.87641	0.13909
3.40	1.74759	0.90211	0.11782
3.60	1.93029	0.92370	0.09802
3.80	2.11692	0.94151	0.08004
4.00	2.30676	0.95592	0.06414
4.20	2.49919	0.96736	0.05042
4.40	2.69365	0.97628	0.03887
4.60	2.88968	0.98309	0.02938
4.80	3.08689	0.98819	0.02177
5.00	3.28499	0.99194	0.01580
5.20	3.48373	0.99464	0.01124
5.40	3.68292	0.99655	0.00784
5.60	3.88244	0.99787	0.00535
5.80	4.08217	0.99876	0.00357
6.00	4.28206	0.99936	0.00233



- Blasius solution calculated using a spreadsheet
- ODEs integrated numerically using Euler Predictor-Corrector method
- $f''(0) = \mathrm{h}(0)$ determined by trial and error ("shooting method") as 0.332.

Blasius Results

- From the solution, we can directly get the velocity components as

$$u = V_\infty f', \qquad v = V_\infty \frac{\eta f' - f}{2\sqrt{Re_x}}$$

- Notice that v is a function of both η and x, and scales with $\frac{1}{\sqrt{Re_x}}$.
- Boundary layer thickness is obtained by noting from the tabulated results that $u = f'V_\infty = 0.99\, V_\infty$ at approximately $\eta = 5.0$.
- Using the definition of η, we can write the boundary layer thickness $\delta(x)$ as the vertical coordinate y where $\eta = 5.0$:

$$\frac{\delta(x)}{x} = \frac{5.0}{\sqrt{Re_x}}$$

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Blasius Results (cont.)

- The displacement and momentum thicknesses can be integrated (numerically) from the velocity profile:

$$\delta^* = \int_0^\infty \left(1 - \frac{u}{V_\infty}\right) dy = \sqrt{\frac{\nu x}{V_\infty}} \int_0^\infty (1 - f') \, d\eta = 1.72 \sqrt{\frac{\nu x}{V_\infty}}$$

$$\theta = \int_0^\infty \frac{u}{V_\infty}\left(1 - \frac{u}{V_\infty}\right) dy = \sqrt{\frac{\nu x}{V_\infty}} \int_0^\infty f'(1 - f') \, d\eta = 0.664 \sqrt{\frac{\nu x}{V_\infty}}$$

or

$$\frac{\delta^*(x)}{x} = \frac{1.72}{\sqrt{Re_x}} \qquad \frac{\theta(x)}{x} = \frac{0.664}{\sqrt{Re_x}}$$

Blasius Results (cont.)

- Wall shear stress is obtained from the velocity gradient at the wall $f''(0) = 0.332$:

$$\tau_W = \mu \frac{\partial u}{\partial y}\Big|_{y=0} = 0.332 \frac{\mu V_\infty}{\sqrt{\nu x / V_\infty}}$$

- The friction coefficient C_f can be computed as follows:

$$C_f = \frac{\tau_W}{\frac{1}{2}\rho_\infty V_\infty^2} = \frac{0.664}{\sqrt{Re_x}} = \frac{\theta}{x}$$

- The drag coefficient on a plate of length L is:

$$C_D(L) = \frac{1}{L}\int_0^L C_f \, dx = 2C_f(L) = \frac{1.328}{\sqrt{Re_L}}$$

drag on one side of the plate

Comparison with Integral Analysis Approach

- Finally, let us compare results from the integral approach with those from the exact Blasius solution:

$$\frac{\delta}{x} = \frac{5.48}{\sqrt{Re_x}} \qquad \frac{\delta^*}{x} = \frac{1.83}{\sqrt{Re_x}} \qquad \frac{\theta}{x} = \frac{0.73}{\sqrt{Re_x}} \qquad C_D = \frac{1.46}{\sqrt{Re_L}}$$

Approximate solution from integral analysis

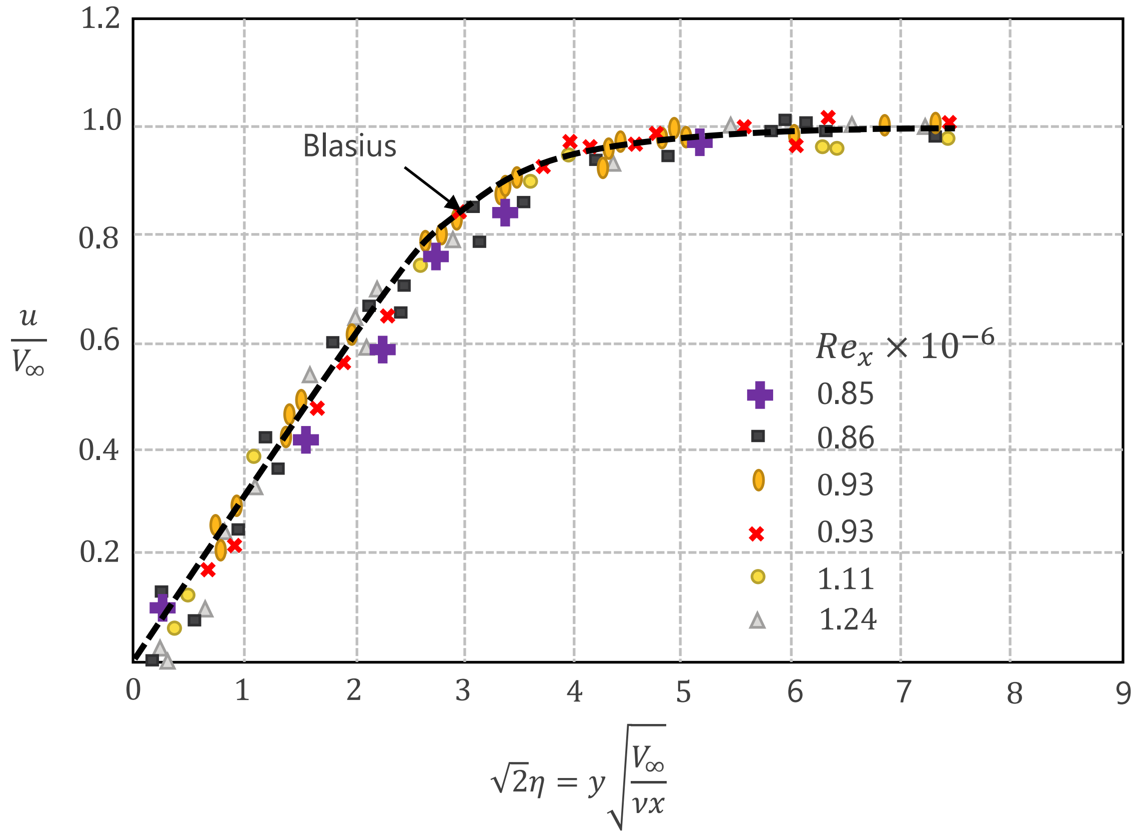
$$\frac{\delta}{x} = \frac{5.0}{\sqrt{Re_x}} \qquad \frac{\delta^*}{x} = \frac{1.72}{\sqrt{Re_x}} \qquad \frac{\theta}{x} = \frac{0.664}{\sqrt{Re_x}} \qquad C_D = \frac{1.328}{\sqrt{Re_L}}$$

Exact Blasius solution

- As was stated earlier, the integral analysis values are within 10% of the exact solution. Unlike the Blasius solution they were obtained without complicated math by examining the boundary layer flow from the first principles of conservation laws.

Comparison with Experiments

- Figure below compares the profile $f' = u/V$ with the experimental results obtained by Liepmann (1943).



Summary

- In this lesson we discussed the solution methodology of the boundary layer equations originally proposed by Blasius.
- This solution covers a wide range of laminar boundary layer flows from $Re = 1000$ to 10^6.
- The idea of the similarity solution propose by Blasius can be extended to other types of flows including turbulent jets and wakes.
- Next, we will discuss some extensions of the 2D Blasius flat plate solutions to other configurations, will briefly cover 3D laminar boundary layer theory, and consider other approaches to analyze boundary layers.

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