Dimensionless Parameters

Dimensional Analysis and Similarity – Lesson 2

Non-Dimensional Incompressible Navier –Stokes Equations

- To illustrate the non-dimensionalization process, let us consider the incompressible Navier-Stokes equations shown below (ignoring the body force term):

$$\nabla \cdot \vec{V} = 0$$

$$\rho\left[\frac{\partial u}{\partial t} + \nabla \cdot (\vec{V} u)\right] = -\frac{\partial p}{\partial x} - \mu \nabla^2 u$$

$$\rho\left[\frac{\partial v}{\partial t} + \nabla \cdot (\vec{V} v)\right] = -\frac{\partial p}{\partial y} - \mu \nabla^2 v$$

$$\rho\left[\frac{\partial w}{\partial t} + \nabla \cdot (\vec{V} w)\right] = -\frac{\partial p}{\partial z} - \mu \nabla^2 w$$

$$\rho C_p\left[\frac{\partial T}{\partial t} + \nabla \cdot (\vec{V}\, T)\right] = k\nabla^2 T + (\bar{\bar{\tau}} \cdot \nabla)\vec{V}$$

Dimensionless Variables

- Let us introduce the following non-dimensional variables into the equations:
  - Time: $t^* = t/t_0 = (tV_0)/L_0$
  - Space: $x^* = x/L_0, y^* = y/L_0, z^* = z/L_0$
  - Density: $\rho^* = \rho/\rho_0$
  - Velocity: $u^* = u/V_0, v^* = v/V_0, w^* = w/V_0$
  - Pressure: $p^* = p/p_0 = p/\rho_0 V_0^2$
  - Viscosity: $\mu^* = \mu/\mu_0$
  - Temperature: $T^* = T/T_0$
  - Specific Heat: $C_p^* = C_p/C_{p0}$
  - Thermal conductivity: $k^* = k/k_0$
- We make use of the reference quantities (denoted by the 0 subscript) in the above, which are constants that depend on the problem at hand. For example, the freestream flow conditions can be used for external flows.
  - Note the use of L_0/V_0 as a time reference parameter, and $\rho_0 V_0^2$ as a pressure reference parameter.

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Non-Dimensional Continuity and Momentum Equations

- Substitution of the non-dimensional variables into the continuity and momentum equations yields:

$$\nabla \cdot \vec{V}^* = 0$$

$$\rho^* \left[\frac{\partial u^*}{\partial t^*} + \nabla \cdot \left(\vec{V}^* u^* \right) \right] = -\frac{\partial p^*}{\partial x^*} + \frac{1}{Re} \nabla^2 u^*$$

$$\rho^* \left[\frac{\partial v^*}{\partial t^*} + \nabla \cdot \left(\vec{V}^* v^* \right) \right] = -\frac{\partial p^*}{\partial y^*} + \frac{1}{Re} \nabla^2 v^*$$

$$\rho^* \left[\frac{\partial w^*}{\partial t^*} + \nabla \cdot \left(\vec{V}^* w^* \right) \right] = -\frac{\partial p^*}{\partial z^*} + \frac{1}{Re} \nabla^2 w^*$$

- This process results in a non-dimensional form of the equations where the diffusion terms involve a dimensionless parameter: the Reynolds number.

$$Re = \frac{\rho_0 V_0 L_0}{\mu_0}$$

Reynolds Number

- The Reynolds number is the fundamental non-dimensional parameter in all viscous fluid flows, and it is commonly denoted as *Re*:

$$Re = \frac{\rho_0 V_0 L_0}{\mu_0}$$

- Physically, it represents the ratio of inertial forces to viscous forces acting on the fluid.
- The Reynolds number is used as an indicator of whether the flow is laminar or turbulent:
  - Flows with low *Re* are dominated by viscosity and remain laminar.
  - In flows with high *Re*, viscosity effects are limited to very narrow regions along walls (call boundary layers), or along boundaries of different flow streams (called shear layers). Such flows are always highly turbulent.

Non-Dimensional Incompressible Energy Equation

- We can apply a similar non-dimensionalization to the incompressible energy equation, with dimensionless temperature, specific heat and thermal conductivity as follows:

$$\rho^* C_p^* \left[\frac{\partial T^*}{\partial t^*} + \nabla \cdot \left(\vec{V}^* T^*\right)\right] = \frac{k^*}{RePr} \nabla^2 T^* + \frac{Br}{RePr} \Phi^*$$

where Φ is the viscous dissipation term.

- The two new non-dimensional numbers in the above equation are:

$$Pr = \frac{C_{p0}\mu_0}{k_0}$$

Prandtl Number

$$Br = \frac{V_0^2 \mu_0}{k_0 T_0}$$

Brinkman Number

The Prandtl and Brinkman Numbers

- The parameter Pr is called the Prandtl Number and it represents the ratio of viscous diffusion (of momentum) to thermal diffusion in the fluid.
  - The product of the Reynolds number and Prandtl number appears in the thermal diffusion term. This product is sometimes defined as the Peclet Number (Pe).

- The parameter Br is called the Brinkman Number and represents the ratio of heat generated due to viscous dissipation to the heat exchanged by conduction.
  - The ratio of the Brinkman number and Prandtl number appears in the viscous dissipation term. This ratio is sometimes defined as the Eckert number (Ec).

$$Ec = \frac{Br}{Pr} = \frac{V_0^2}{C_{p0} T_0}$$

Eckert Number

Note: The reference temperature T_0 is often replaced with a wall-freestream temperature difference $\Delta T = T_w - T_\infty$, as this leads to a more meaningful representation of the dissipated thermal energy in high speed flows.

Other Common Non-dimensional Numbers

- **Froude number (Fr)** defines the ratio of inertial forces to gravitational body forces in the fluid. Dating back to the original work done by William Froude, the Froude number is customarily defined as a velocity-length ratio:

$$Fr = \frac{V_0}{\sqrt{g_0 L_0}}$$

  - ❖ Froude number is important in naval architecture, as the resistance of a ship's hull correlates to the Froude number.

- **Rayleigh number (Ra)** is used to characterize buoyancy-driven flows (natural convection). The magnitude of the Rayleigh number is a good indication of whether the natural convection boundary layer is laminar or turbulent.

$$Ra = \frac{\rho \beta \Delta T L^3 g}{\mu \alpha}$$

β – thermal expansion coefficient
ΔT – temperature difference across distance L
α – thermal diffusion coefficient

- **Mach number (M)** is the ratio of characteristic flow velocity to the speed of sound in compressible flows. It is used to describe different regimes of a compressible flow: subsonic, transonic and supersonic.

- There are many other non-dimensional numbers in fluid dynamics specific to different physical models. We are not going to describe them all here. Instead, we will be introducing these non-dimensional parameters as the need arises.

Similarity Observations

- The non-dimensional forms of the Navier-Stokes equations are essentially similar to the dimensional forms, with the addition of the dimensionless parameters in specific terms, e. g., Reynold’s number, Prandtl number, etc.

- Dynamic similarity can thus be ensured if we set the appropriate dimensionless parameters to consistent values.

- For example, for a geometrically similar experimental model versus full-scale, Reynolds numbers must be the same in both models to have dynamic similarity. This can be achieved by:
  - Using a fluid with a different density and viscosity for the experiment.
  - Conducting the experiment with a different reference velocity.

    Using a 1/10th scale model of a car body in a wind tunnel (assuming ambient air properties), we impose an inlet flow velocity that is 10 times bigger than the real-world vehicle.

Summary

- Dynamic similarity is the cornerstone of experimental fluid mechanics.
- By matching dimensionless groups, we can match the observed behavior of fluid flows for a model system with the flow expected for a full-sized system.
- Dimensionless groups can be obtained by non-dimensionalizing the governing equations of fluid mechanics.
- Non-dimensional equations are useful for scaling various physical phenomena and help us develop simplified models of fluid behavior for specific classes of problems, e. G., inviscid flows, laminar viscous dominated flows, etc.

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